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PROPOSITIONAL CALCULI BASED ON SUBRESIDUATION

Consider pairs (A, Q), where A is a distributive lattice with $0, 1$ and Q is a sublattice of A containing $0, 1$ such that for each $x, y \in A$ there is a largest $z \in Q$ (denoted by $x \to^Q y$) satisfying $xz \leq y$. We may define systems of propositional calculi in which $p \,\&\, q$, $p \vee q$, $p \supset q$, $\sim p$, $\Box p$ are interpreted by pq, $p + q$, $p \to^Q q$, $p \to^Q 0$, $1 \to^Q p$, respectively, as well as fragments of such systems. The valid formulas are those formulas which are identically 1 in this interpretation.

This framework provides a unified method of classifying many known systems. Excluding language semantics, the formulas valid in all pairs (A, Q) are exactly the theorems of Lewy $S4$ (the $S4$ theorems which involve strict implication and strict negation). The system Lewy $S5$ is characterized by all pairs (A, Q) in which Q is a subalgebra of the Boolean algebra B of complemented elements of A. The full systems $S4$ or $S5$ with strict implication are characterized by pairs (A, Q) in the same way as the corresponding Lewy systems, except that A is required to be a Boolean algebra. The intuitionist propositional calculus is characterized by pairs (A, Q) with $Q = A$. Other examples can be given.

Axioms for the first four systems mentioned above were given by I. Hacking (JSL, vol. 28, 1963). We focus attention on the relation of Q with respect to the underlying Boolean algebra B of A. The formulas valid in all pairs (A, Q) where Q contains B turns out to be the same as the theorems of Lewy $S4$; this also holds for all pairs (A, Q) where Q is contained in B (with Q not necessarily a subalgebra of B). A characteristic axiom for the system Lewy $S5$ is a law of included middle: $\Box p \vee \sim (\Box p \vee \sim p) \vee \sim p$. The case $Q = B$ is properly stronger than Lewy $S5$. A characteristic axiom for the formulas valid in all pairs (A, B) is $\Box p \vee \sim [\Box(p \vee q) \& \sim (p \,\&\, q)] \vee \sim p$, from which the previous law of included middle follows by substitution. Examples of such systems are Post algebras and the BL algebras described in our previous abstract (JSL, vol. 38, 1973).