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ON CARDINALITY OF MATRICES STRONGLY ADEQUATE FOR THE INTUITIONISTIC PROPOSITIONAL LOGIC

Gödel [2] stated that there is no finite matrix adequate for the intuitionistic propositional logic (INT). However, a denumerable adequate matrix was found by Jaśkowski [5]. In this paper it is shown that no denumerable matrix is strongly adequate for INT which was previously conjectured by prof. R. Suszko.

Let $\mathcal{F} = \langle F, \wedge, \vee, \rightarrow, \neg \rangle$ be the free algebra in the class of all algebras of the similarity type $\langle 2, 2, 2, 1 \rangle$ free-generated by a denumerably infinite set $V \subseteq F$. The elements of F are called formulas and denoted by $\alpha, \beta, \ldots$, the elements of V are called variables and denoted by $x, y, \ldots$. The familiar abbreviation $\alpha \leftrightarrow \beta$ is used for $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. The symbol Sb denotes the consequence operation in F determined by the substitution rule, and Cn denotes the consequence operation in F determined by the set of theorems of INT (see [3]) and the detachment rule. If $X \subseteq F$ then $\equiv_X$ denotes the congruence relation of $\mathcal{F}$ given by the condition: $\alpha \equiv_X \beta$ iff $\alpha \leftrightarrow \beta \in Cn(X)$. By an intermediate logic we mean a set of formulas Δ such that $\Delta = Cn(Sb(\Delta)) \neq F$. The consequence operation of the intermediate logic Δ is denoted by Cn_Δ ($Cn_\Delta(X) = Cn(\Delta \cup X)$).

Any pair $\langle \mathcal{M}, D \rangle$ where $\mathcal{M}$ is an algebra similar to $\mathcal{F}$ and D is a subset of the domain of $\mathcal{M}$ is called a matrix. $C_{\langle \mathcal{M}, D \rangle}$ denotes the consequence operation of the matrix $\langle \mathcal{M}, D \rangle$ ($\alpha \in C_{\langle \mathcal{M}, D \rangle}(X)$ iff $h(X) \subseteq D$ implies that $h(\alpha) \in D$ for every homomorphism $h : \mathcal{F} \mapsto \mathcal{M}$). For the sake of simplicity we use the symbol $E\langle \mathcal{M}, D \rangle$ instead of $C_{\langle \mathcal{M}, D \rangle}(\emptyset)$ for denoting the content of the matrix $\langle \mathcal{M}, D \rangle$. A matrix $\langle \mathcal{M}, D \rangle$ is said to be adequate (strongly adequate) for an intermediate logic Δ iff $E\langle \mathcal{M}, D \rangle = \Delta$ ($C_{\langle \mathcal{M}, D \rangle} = Cn_\Delta$). By a general theorem of Łoś, Suszko [6] (see also corrections in Wójcicki

[10]) we know that every intermediate logic has a strongly adequate matrix of the power not exceeding $2^{\aleph_0}$. The following lemma shoves that investigating matrices strongly adequate for intermediate logics we can confine ourselves to examining only so called pseudo-Boolean algebras (see [8]).

LEMMA 1. *If Δ is an intermediate logic and $\langle \mathcal{M}, D\rangle$ a matrix strongly adequate for Δ then the following conditions hold:*

(i) The relation $\equiv_D$ such that $a \equiv_D b$ iff $a \leftrightarrow_{\mathcal{M}} b \in D$ is a congruence of the algebra $\mathcal{M}$;
(ii) $\mathcal{M}/\equiv_D$ is a pseudo-Boolean algebra having D as unit-element;
(iii) $\langle \mathcal{M}/\equiv_D, D/\equiv_D\rangle$ is a matrix strongly adequate for Δ.

K denotes the class of all pseudo-Boolean algebras. The German capitals: $\mathcal{A}$, $\mathcal{B}$, ... denote algebras from K and the corresponding Latin capitals: $A, B, \dots$ their domains. Every algebra $\mathcal{A} \in K$ in a natural way can be considered as a matrix $\langle \mathcal{A}, \{1_{\mathcal{A}}\}\rangle$ where $1_{\mathcal{A}}$ denotes the unit-element of $\mathcal{A}$. Thus, for the sake of simplicity, the symbols $C_{\mathcal{A}}$ $E\mathcal{A}$ will be used to abbreviate $C_{\langle \mathcal{A}, \{1_{\mathcal{A}}\}\rangle}$ and $E\langle \mathcal{A}, \{1_{\mathcal{A}}\}\rangle$ respectively. $\mathcal{A}$ is said to be strongly compact (see [8]) iff there exists the greatest element in $A - \{1_{\mathcal{A}}\}$ with respect to lattice ordering $\leqslant_{\mathcal{A}}$ of the algebra $\mathcal{A}$. Such an element – if it exists – will be denoted by $\star_{\mathcal{A}}$. The symbol K_0 denotes the class of all denumerable strongly compact algebras from K. With an intermediate logic Δ one can associate in an one-to-one manner an equational class of algebras $K(\Delta) = \{\mathcal{A} : \mathcal{A} \in K, \Delta \subseteq E\mathcal{A}$ and also in such a manner a class of denumerable strongly compact algebras $K_0(\Delta) = K(\Delta) \cap K_0$. In the sequel it will be convenient to have the symbol $\mathcal{A}\oplus$ for denoting the result of applying Jaśkowski's Γ-operation (see [5]) to the algebra $\mathcal{A}$ and the symbol $\equiv_a$ for the congruence relation in $\mathcal{A}$ determined by an element $a \in A$ ($b \equiv_a c$ iff $a \leqslant_{\mathcal{A}} b \leftrightarrow_{\mathcal{A}} c$).

LEMMA 2. *For every $\mathcal{A} \in K$ the following conditions hold:*

(i) $\mathcal{A}\oplus$ is strongly compact;
(ii) $\mathcal{A}\oplus, \mathcal{BL}\oplus$ are isomorphic iff so are $\mathcal{A}$ and $\mathcal{B}$;
(iii) If $G \subseteq A$ generates $\mathcal{A}$ then $G \cup \{\star_{\mathcal{A}\oplus}\}$ generates $\mathcal{A}\oplus$.

Suppose that for each denumerable algebra $\mathcal{A} \in K$ we are given a fixed one-to-one mapping $f_{\mathcal{A}} : A \mapsto V$ (for the sake of simplicity the variable $f_{\mathcal{A}}(a)$ will be denoted by z_a). Generalizing an idea of Jankov [4] we define the diagram of a denumerable $\mathcal{A} \in K$ putting:

$$\begin{array}{lll} DG(\mathcal{A}) & = \{(z_a \wedge z_b) \leftrightarrow z_{a \wedge_{\mathcal{A}} b} & : a, b \in A\} \cup \\ & \cup \{(z_a \vee z_b) \leftrightarrow z_{a \vee_{\mathcal{A}} b} & : a, b \in A\} \cup \\ & \cup \{\neg z_a \leftrightarrow z_{\neg_{\mathcal{A}} a} & : a \in A\} \end{array}$$

Lemma 3. (comp. [4]). *For every $\mathcal{A} \in K_0$ and $\mathcal{L} \in K$ the following conditions are equivalent:*

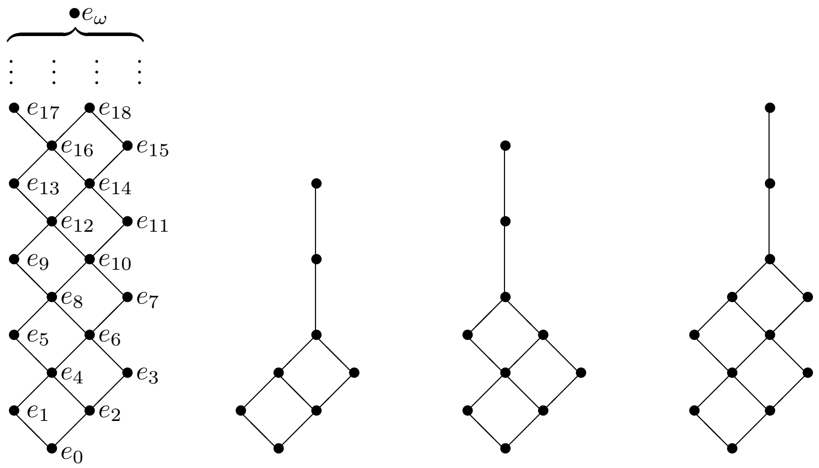
(i) $\mathcal{A}$ is embeddable into $\mathcal{L}$;
(ii) $z_{\star_{\mathcal{A}}} \notin C_{\mathcal{L}}(DG(\mathcal{A}))$.

An immediate consequence of Lemma 3 is the following useful criterion of the strong adequacy:

Theorem 1. *If Δ is an intermediate logic and $\mathcal{L} \in K$ then the following conditions are equivalent:*

(i) $\mathcal{B}$ is strongly adequate for Δ;
(ii) $\mathcal{B} \in K(\Delta)$ and every algebra from $K_0(\Delta)$ is embeddable into $\mathcal{B}$.

Let φ_i $(i = 0, 1, \ldots, \omega)$ be the formulas defined as follows: $\varphi_0 = (y \wedge \neg y)$, $\varphi_1 = (\neg y)$, $\varphi_2 = (y)$, $\varphi_3 = (\neg\neg y)$, $\varphi_4 = (y \vee \neg y)$, $\varphi_{2i+5} = (\varphi_{2i+3} \rightarrow \varphi_{2i+2})$, $\varphi_{2i+6} = (\varphi_{2i+3} \vee \varphi_{2i+1})$, $\varphi_\omega = (y \rightarrow y)$. Rieger [9] and then (11 years later) Nischimura [7] proved that the equivalence classes of the formulas φ_i, $i = 0, 1, \ldots, \omega$ under the congruence $\equiv_{INT}$ are pairwise distinct and form a subalgebra of the quotient algebra $\oint / \equiv_{INT}$. Obviously such a subalgebra is free in K with the one-element free-generating set. To the honour of the first finder let us denote it by $\mathcal{R}$ and illustrate by the following picture $(d.\infty)$ (e_i denotes the equivalence class of φ_i):



$(d.\infty)$ $(d.0)$ $(d.1)$ $(d.2)$

Putting ϑ_i $\vartheta_i = (\mathcal{R}/\equiv_{e_{2i+7}})\oplus$ for $i = 0, 1, \ldots$ one obtains the sequence of algebras examined in Gerčiu, Kuznecov [1]. The pictures: $(d.0)$, $(d.1)$ and $(d.2)$ visualize the lattice orderings of ϑ_0, ϑ_1, ϑ_2 respectively. For every algebra ϑ_i we have the corresponding formula $\delta_i = \varphi_{2i+7} \to (x \vee (x \to \varphi_{2i+6}))$ (the two variables x and y occurring in δ_i are supposed to be distinct) and for every set of natural numbers $I \subseteq \omega$ the corresponding intermediate logic $L(I) = Cn(Sb(\delta_i : i \in I))$. Let us quote the following lemma due to Gerčiu, Kuznecov [1]:

Lemma 4. (see [1]).

(i) *If $i \neq j$ then there is no embedding of ϑ_i into a quotient algebra of ϑ_j;*

(ii) *for every $\lfloor \in K$, $\delta_i \in E\lfloor$ iff there is no embedding of ϑ_i into a quotient algebra of $\lfloor$;*

(iii) $\delta_i \notin L(\omega - (i))$;

(iv) $L(I) = L(J)$ *iff* $I = J$.

Let $\mathcal{F}_2 = \langle F_2, \wedge, \vee, \to, \neg \rangle$ be the subalgebra of $\mathcal{F}$ generated by the variables: x and y. As a simple corollary of Lemma 4 (iv) we have the following:

Lemma 5. *The algebras:* $\mathcal{F}_2/\equiv_{L(I)}$, $\mathcal{F}_2/\equiv_{L(J)}$ *are isomorphic iff* $I = J$.

Theorem 2. *The number of non-isomorphic algebras in* K_0 *having a three-element generating set is* $2^{\aleph_0}$.

Combining Theorem 1, 2 and Lemma 1 we obtain the following:

Theorem 3. *There is no denumerable matrix strongly adequate for* INT.

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