APPLICATION OF THE THEORY OF LOGICAL MATRICES IN THE INDEPENDENCE PROOFS
(on the basis of M. Wajsberg’s work [8])

In 1930 Łukasiewicz and Tarski introduced in “Untersuchungen über den Aussagenkalküll”, the very important concept of logical matrix and also the concept of normal matrix which is a basis for the Wajsberg’s paper [8] written in Lomża in 1934 and published two years latter.

The aim of Wajsberg’s work is to describe some systematic procedure for proving independence or more exactly independence of some sets of formulas built up in the usual way from propositional variables by means of the implication connective and the negation connective. To realize his intention Wajsberg indicate certain types of matrices having relatively uncomplicated decision procedures for their contents.

**Definition.** The matrix is a quadruple $M = (A(M), B(M), C_M, N_M)$ such that $A(M)$, $B(M)$ are disjoint and non-empty sets, $C_M$ is a binary operation in the set $W(M) = A(M) \cup B(M)$, $N_M$ is an unary operation in $W(M)$, and the following condition of normality holds:

If $x \in B(M)$, $y \in A(M)$ then $C_M xy \in A(M)$.

The definition above differs from the Tarski’s definition of normal matrix only by the requirement of non-emptiness for $A(M)$ and $B(M)$ which is dropped in the Tarski’s definition. Wajsberg says that the matrix has a degree $m$ ($m$ is cardinal number) iff the set $W(M)$ has the cardinality $m$.

The first family of matrices considered in [8] consists of so called congruence matrices.

A matrix $M$ is called congruence matrix iff it satisfies the following conditions


For this matrices Wajsberg proved (using some arithmetical statements) the following two theorems

**Theorem.** For every matrix whose degree is a prime number there exists an isomorphic congruence matrix.

**Theorem.** For every finite matrix there exists an equivalent (having the same content) congruence matrix whose degree is a prime number.

According to Wajsberg especially important kind of congruence matrices are that called by him sum-matrices.

A sum matrix of the order \(i\) \((i \in \mathbb{Z})\) is a congruence matrix \(M\) with the following properties

(I) \(W(M) = \{j \mid j \in \mathbb{Z}, 0 < j < m \in \mathbb{Z}\}\)

(II) \(B(M) = \{0\}\)

(III) \(C_Mxy = [x + y]_m, N_Mx = [ix]_m\)

**Theorem.** Let \(M\) be a sum matrix of the order \(i\) and the degree \(m\). Then the following conditions are equivalent

(I) A formula \(\alpha\) belongs to the content of \(M\)

(II) The number of occurrences of each propositional variable in \(\alpha\) is divisible by \(m\) (each occurrence after \(k\) signs of negation should be counted as \(i^k\) occurrences).

Wajsberg gives further theorems:

**Theorem.** If \(m\) is a common divisor of all the numbers of occurrences of variables in the formulas from the set \(X\) then \(m\) has this property also with respect to formulas from \(Cn(X)\).

**Theorem.** Every formula containing only one variable and not being a single variable is valid in a certain sum matrix.

The following remarkable property of formulas built of one variable results from theorem above
Theorem. The set of consequences of formulas built of one variable and not being a single variable is always distinct from the set of all formulas.

The concept of sum-matrix is a specification of some more general concept of linear congruence matrix, i.e. congruence matrix such that

\[ C_M xy = [ax + by + c]_m \quad N_M x = [dx + e]_m \]

In order to guarantee the normality condition for a linear congruence matrix one should require that the coefficients \( a, b, c \) satisfy following

If \( x, z \in B(M) \) and \( z \equiv ax + by + c \) (modulo \( m \)), then \( y \in B(M) \).

To construct a linear congruence matrix satisfying a given formula \( \alpha \) one should build so called the normal linear form of \( \alpha \), i.e.,

\[ L(\alpha) = Pp + Qq + \ldots + Rr + Z \]

where every coefficient \( P, Q, \ldots, R, Z \) is a polynomial of \( a, b, c, d, e \). The procedure of constructing a linear congruence matrix satisfying \( \alpha \) is given by the following theorem

Theorem. A formula \( \alpha \) is valid in a linear congruence matrix \( M \) iff all the coefficients of its linear normal form \( L(\alpha) \) are equivalent to 0 modulo \( m \) and furthermore \( [Z]_m \in B(M) \).

Next Wajsberg takes into consideration infinite matrices and define so called equality matrices by the following conditions

(I) \( W(M) \) is a set of algebraic numbers and \( B(M) \) is a finite subset of \( W(M) \).

(II) \( C_M xy \) and \( N_M x \) are rational functions.

The definition above allows Wajsberg to prove the following

Theorem. Let \( M \) be a rational equality matrix (i.e. equality matrix in which \( W(M) \) is the set of rational numbers). Then the decision problem for the content of \( M \) is solvable.

By a linear matrix Wajsberg means an equality matrix in which

\[ C_M xy = ax + by + c \quad N_M x = dx + e \quad \text{and} \quad B(M) \] is an one-element set.

In order to guarantee the normality for a linear matrix one should require that the coefficients \( a, b, c \) satisfy one of following three conditions

(I) \( b = 0, a = 1, c \neq 0 \)

(II) \( b \neq 0, a + b - 1 = 0 \) for every \( B(M) \)

(III) \( b \neq 0, a + b - 1 \neq 0 \)

\[ B(M) = \{ \frac{c}{a + b - 1} \} \]
The procedure of constructing a linear matrix satisfying a given formula is similar to that described previously for linear congruence matrices.

The following theorems give useful characterization of contents of linear matrices.

**Theorem.** If the coefficients $a$ and $b$ of the linear matrix $M$ are such that $a = 0$ or $b = 0$ then every strictly implicational formula not being a single variable belongs to the content of $M$.

**Theorem.** The content of the linear matrix $M_k$ such that $C_{M_k}xy = x + 1$, $N_{M_k}x = 0$, $B(M_k) = \{k\}$ is equal to the set of all formulas beginning with $C_{-\infty}CN -$.

**Theorem.** The content of the linear matrix $M$ such that $C_Mxy = y$, $N_Mx = 0$, $B(M) = \{0\}$ is equal to the set of all formulas having one of the forms $N\alpha$ or $C\alpha_1 \ldots C\alpha_i N\beta$.

**Theorem.** The content of the linear matrix $M_k$ such that $W(M_k) = \mathbb{N}$, $B(M_k) = \{0, \ldots, k\}$, $C_{M_k}xy = y + 1$, $N_{M_k}x = 0$ is equal to the set of all formulas having one of the forms $N\alpha$ or $C\alpha_1 \ldots C\alpha_i N\beta$ where $i \leq k$.

**Theorem.** The content of the linear matrix $M$ such that $W(M) = \mathbb{Z}$, $B(M) = \{0\}$, $C_Mxy = (x + 1)y^2$, $N_Mx = -1$ is equal to the set of all formulas of the form $C\alpha_1 \ldots C\alpha_n CN\beta\gamma$.

The last family of matrices considered by Wajsberg consists of so called by him conditional matrices, i.e. matrices in which the operations $C_M$ and $N_M$ are defined by means of conditions connected with some partition of the set $W(M)$. Wajsberg gives several examples of those matrices. The well-known matrices of Łukasiewicz are the most important example of this kind. Among conditional matrices Wajsberg distinguishes so called interval matrices in which conditions defining $C_M$ and $N_M$ are connected with intervals, sums of intervals or intersections of the set $W(M)$.

The discussed paper of Wajsberg preserved its actuality only in small part, but nevertheless contributed in its time to the development of the theory of logical matrices which nowadays flourished into the leading trend of the modern logic – the theory of models.

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