WAJSBERG ON THE FIRST-ORDER PREDICATE CALCULUS FOR THE FINITE MODELS
(on the basis of Wajsberg’s work [4])

Wajsberg considers the first-order predicate calculus for the finite models in [4]. The author assumes the formalization of the first-order predicate calculus presented in [HA] with the following changes and completions. By “formulas of the first-order predicate calculus” are understood only such formulas which do not contain individual constants, predicate constants nor variable symbols of the propositional calculus. The special role play the formulas in which appear only one-argument predicates, which will be denoted as Cl-formulas here. True formulas in a model containing exactly \( k \) elements (\( k \) is a natural number creator than 0) are called \( k \)-true formulas, while \( k \)-true formulas which are not at the same time \( k + 1 \)-true formulas are described as exactly \( k \)-true formulas. The expression “\( A \) follows from \( B, C, \ldots \)” is equivalent to the expression “\( A \) is provable in the first-order predicate calculus with \( B, C, \ldots \) as new axioms added”.

Wajsberg proves the following theorem:

**THEOREM I.** *From every exactly \( k \)-true formula follows every \( k \)-true formula.*

The proof goes as follows.

**LEMMA 1.** *There is certain exactly \( k \)-true Cl-formula, from which follows every \( k \)-true formula.*

To prove the above lemma Wajsberg constructs exactly \( k \)-true Cl-formula, which has the following schema:

\[
(S_k) \quad A_1 \lor \ldots \lor A_k
\]
where

\[ A_i \ (1 \leq i \leq k) \]

has the following form

\[ F_i(x_i) \rightarrow F_i(x_{i+1}) \lor \ldots \lor F_i(x_{k+1}) \]

For example, for \( k = 2 \) from \( (S_k) \) we get the formula

\[ [F_1(x_1) \rightarrow (F_1(x_2) \lor F_1(x_3))] \lor (F_2(x_2) \rightarrow F_2(x_3)) \]

The proof of Lemma 1 consists in showing that from \( (S_k) \) follows every \( k \)-true formula. While proving it Wajsberg uses only propositional and predicate calculi. The same concerns the proofs of the next lemmas.

**Lemma 2.** From every exactly \( k \)-true \( Cl \)-formula follows every \( k \)-true \( Cl \)-formula.

In order to prove Lemma 2 it is necessary to consider some properties of \( Cl \)-formulas. Wajsberg uses here results due to Herbrand [H].

From each conjunction with \( n \) members

\[ F_1(x) \land \ldots \land F_n(x) \]

can be formed \( 2^n \) of different conjunctions with \( n \) members by negation of any of the expressions \( F_j(x) \ (1 \leq j \leq n) \). These conjunctions are elementary \( n \)-degree conjunctions. Let \( K_i \ (i \leq n) \) denote the elementary \( i \)-degree conjunction. The expressions \( \bigvee_x K_i(x) \) and \( \neg \bigvee_x K_i(x) \) are respectively positive and negative basis \( i \)-degree formula and disjunction of different basis formulas of the equal degree is the normal disjunction of the same degree.

Herbrand proved the following theorem:

Every \( Cl \)-formula without free variables, which has exactly \( n \) predicates can be reduced to the form of the conjunction of the normal \( n \)-degree disjunctions.

Owing to this theorem the proof of Lemma 2 can be reduced to the proof of the fact that every \( k \)-true normal disjunction follows from every exactly \( k \)-true normal disjunction.

The next two lemmas state some structural properties of the \( k \)-true and exactly \( k \)-true normal disjunction necessary for the proof of Lemma 2.
Lemma 2a.

(1) If $A$ is exactly $k$-true normal disjunction then $A$ contains exactly $k + 1$ different basis formulas beginning with negation and no number of $A$ is the negation of any other member.

(2) If $A$ is the $k$-true normal disjunction, then $A$ is the theorem, or contains at least $k + 1$ basic formulas beginning with negation.

Lemma 2b. If for the given formulas $A_1 \ldots A_{2n}$ the following expressions are the theorems

(1) $-(A_i \land A_j)$, $i, j \leq 2^n$, $i \neq j$

(2) $A_1 \lor \ldots \lor A_{2n}$

and $K_1, \ldots, K_{2^n}$ are the elementary conjunctions built up of the predicates $F_1, \ldots, F_n$, then after substitution of $F_l$ ($l \leq n$) by a certain, the same for all $K_1, \ldots, K_{2^n}$, expressions, the $m$-th conjunction ($m \leq 2^n$) will be equivalent to $A_m$.

From Lemma 1 and Lemma 2 follows, that

From every exactly $k$-true formula follows every $k$-true formula.

To finish the proof of Theorem 1 it is enough to point that holds:

Lemma 3. From every exactly $k$-true formula follows certain exactly $k$-true formula.

In the considerations of the predicate calculus for the finite models usually the following is stated.

Theorem II. From every $k$-true formula follows every $k$-true formula.

Wajsberg’s theorem establishes the essential completion of Theorem II. (Com. [HB], p. 120).

References


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