THE DEGREE OF COMPLETENESS OF SOME FRAGMENTS OF THE INTUITIONISTIC PROPOSITIONAL LOGIC

This is a summary of the result reported in June 1973 at the seminar of the Department of Logic of Jagiellonian University held by Professor S. J. Surma in Cracow. The full text with detailed proofs will appear in the forthcoming number of Zeszyty Naukowe Uniwersytetu Jagiellońskiego, seria Prace z Logiki.

Let the symbols: $\Rightarrow$, $\land$, $\lor$, $\lnot$ serve as the well-known connectives (implication, conjunction, disjunction, negation). By $\Psi$ we always mean a set of connectives from the list above containing the implication sign $\Rightarrow$. We shall introduce some notations using $\Psi$ as a parameter and adopt the convention to omit $\Psi$ in the case: $\Psi = \{\Rightarrow, \land, \lor, \lnot\}$. The symbol $FR_\Psi$ denotes the set of formulas built up in the usual way by means of a denumerable infinity of propositional variables $p, q, \ldots$ and the connectives from $\Psi$. The symbol $INT_\Psi$ denotes the set of formulas from $FR_\Psi$ which are provable in the intuitionistic propositional logic and $C_\Psi$ denotes the consequence operation in $FR_\Psi$ determined by $INT_\Psi$. We say that a subset $Y$ of $FT_\Psi$ is $\Psi$-intermediate logic iff $C_\Psi(Y) = Y$. By degree of completeness of $\Psi$-intermediate logic $Y$ we mean the cardinal number of the set of all $\Psi$-intermediate logics containing $Y$. The degree of completeness of $INT$ has shown to be $2^{\aleph_0}$ by Jankov [4]. The same was proved in [7] for $INT_{\{\Rightarrow, \land, \lor\}}$. In the present paper we will show that for every $\Psi$ the degree of completeness of $INT_\Psi$ is $2^{\aleph_0}$. This result seems to be interesting because if $\Psi \notin \Psi$ then every $\Psi$-intermediate logic is finitely approximable by virtue of the well-known theorem of Diego-Popiel (see [1] and [3]) which yields the existence of very natural examples of logics being finitely approximable but undecidable or non-finitely axiomatizable (comp. [2]).
We will use the German capitals $\mathcal{A}, \mathcal{B}, \ldots$ to denote pseudo-Boolean algebras and the corresponding Latin capitals $A, B, \ldots$ to denote the domains of $\mathcal{A}, \mathcal{B}, \ldots$ respectively. The symbols: $\Rightarrow, \land_\mathcal{A}, \lor_\mathcal{A}, \neg_\mathcal{A}$ denote the pseudo-Boolean operations in $\mathcal{A}$ (relative pseudocomplement, meet, join, pseudocomplement) and the symbols: $1_\mathcal{A}, 0_\mathcal{A}, \leq_\mathcal{A}$ denote the unit element, the zero element and the lattice ordering in $\mathcal{A}$. $\mathcal{A}$ is said to be strongly compact (see [6]) iff there exists the greatest element in $\mathcal{A} - \{1_\mathcal{A}\}$ (such an element – if exists – will be denoted by $^*_\mathcal{A}$). By $\Psi$-reduct of $\mathcal{A}$ we mean the reduct obtained by dropping the pseudo-Boolean operations corresponding to the connectives not belonging to $\Psi$. We say that $\mathcal{A}$ is $\Psi$-generated by the set $G \subseteq \mathcal{A}$ iff $G$ generates $\Psi$-reduct of $\mathcal{A}$ into $\Psi$-reduct of $\mathcal{L}$ will be called $\Psi$-embedding of $\mathcal{A}$ into $\mathcal{L}$.

Lemma 1. Every $\{\Rightarrow\}$-embedding of $\mathcal{A}$ into $\mathcal{L}$ is also $\{\Rightarrow, \land, \lor\}$-embedding of $\mathcal{A}$ into $\mathcal{L}$.

Let us denote by $E_\Psi(\mathcal{A})$ the set of all formulas from $FR_\Psi$ which are associated to $1_\mathcal{A}$ by every valuation of the propositional variables in $\mathcal{A}$. It is obvious that for every non-degenerate $\mathcal{A}$, $E_\Psi(\mathcal{A})$ is $\Psi$-intermediate logic. Unfortunately, not every $\Psi$-intermediate logic is equal to $E_\Psi(\mathcal{A})$ for some $\mathcal{A}$. Take for example $C_{\Rightarrow}(B_0) = (p \Rightarrow q) \lor(p \Rightarrow q) \Rightarrow q)$. Then $C_{\Rightarrow}(B_0) \neq E_{\Rightarrow}(\mathcal{A})$ for every $\mathcal{A}$ because the formula $B_1 = (p \Rightarrow q) \lor(q \Rightarrow p)$ can be proved not to belong to $C_{\Rightarrow}(B_0)$ but one can show that $B_1 \in C(B_0)$ which gives that $B_1 \in E_{\Rightarrow}(\mathcal{A})$ for every $\mathcal{A}$ such that $B_0 \in E_{\Rightarrow}(\mathcal{A})$ (comp. [5]). For this reason $\Psi$-intermediate logic being equal to $E_\Psi(\mathcal{A})$ for some $\mathcal{A}$ will be called ordinary. Notice that every $\{\Rightarrow, \land, \lor\}$-intermediate logic is ordinary.

Having a fixed well-ordering of $FR$ one can introduce the abbreviation $Y \Rightarrow \beta$ ($Y$ finite subset of $FR$, $\beta \in FR$) in order to denote the formula $\beta_0 \Rightarrow \ldots(\beta_n \Rightarrow \beta)$) where $\beta_0, \ldots, \beta_n$ are all the elements of $Y$ in the succession determined by the well-ordering chosen.

Suppose we are given a finite and strongly compact $\mathcal{A}$. Following Jankov [3] we pick a one-to-one mapping $g$ of $A$ into the set of propositional variables and define $\Psi$-characteristic formula of $\mathcal{A}$ putting:
\[ \chi_{\Psi}(A) \{ (g_x \otimes g_y) \Rightarrow g_{x \otimes A y} : \otimes \in \Psi - \{=\}, x, y \in A \} \cup \\
\cup \{ g_{x \otimes A y} \Rightarrow (g_x \otimes g_y) : \otimes \in \Psi - \{=\}, x, y \in A \} \cup \\
\cup \{ g_{\otimes A x} \Rightarrow g_{\otimes A x} : \otimes \in \Psi \cap \{=\}, x \in A \} \cup \\
\cup \{ g_{A A} \Rightarrow g_{A A} : \otimes \in \Psi \cap \{=\}, x \in A \} \Rightarrow \star_{A} g_{A A} \]

In the definition above $\otimes_{A}$ denotes the pseudo-Boolean operation in $A$ corresponding to the connective $\otimes$. Let us notice that $\chi_{\Psi}(A) \in FR_{\Psi}$ and $\chi_{\Psi}(A) \notin E_{\Psi}(A)$ (take $g^{-1}$ as a valuation).

**Lemma 2** (Jankov [3]). Let $A$ be finite and strongly compact. Then the following conditions are equivalent:

(i) $\chi_{\Psi}(A) \notin E_{\Psi}(L)$;

(ii) $E_{\Psi}(L) \subseteq E_{\Psi}(A)$;

(iii) $A$ is $\Psi$-embeddable into some quotient algebra of $L$.

Observe that $C_{\Psi}(\chi_{\Psi}(A)) = \bigcap (C_{\Psi}(Y) : Y \subseteq FR_{\Psi}, Y \nsubseteq E_{\Psi}(A))$ i.e. $C_{\Psi}(\chi_{\Psi}(A))$ is the smallest $\Psi$-intermediate logic not contained in $E_{\Psi}(A)$.

Let us visualize the sequence $\{A_i : i = 0, 1, \ldots\}$ of pseudo-Boolean algebras by means of the following diagrams:
Let us mention that a more exact definition of the sequence \( \{A_i, Li = 0,1,\ldots\} \) can be given in terms of generalized sum of pseudo-Boolean algebras (see [8]). The following observations about the sequence \( \{A_i : i = 0,1,\ldots\} \) are very important for our considerations.

**Lemma 3.**

(i) \( A_i \) is \( \{\Rightarrow, \wedge \} \) generated by the set \( G_i = \{1, 2, a\} \cup \{2i + 11, 2i + 12, b\} \cup \{2j + 7 : j = 0,\ldots,i\} \cup \{*\} \);

(ii) If \( \Gamma \) is a non-trivial filter in \( A_i \) then \( A_j \) cannot be \( \{\Rightarrow\} \)-embedded into \( A_i/\Gamma \) for every \( j = 0,1,\ldots \);

(iii) If \( L \) is a subalgebra of \( A_i \) and \( \{2j + 7 : j = 0,\ldots,i\} \not\subseteq \) then \( A_j \) cannot be \( \{\Rightarrow\} \)-embedded into \( L \) for every \( j = 0,1,\ldots \);

(iv) If \( i \neq j \) then \( A_i \) cannot be \( \{\Rightarrow\} \)-embedded into some quotient algebra of \( A_j \).
In order to simplify the notation let us denote the formula \( X_{\Rightarrow}(A_i, \ \ i = 0, 1, \ldots) \) by \( \alpha_i \) and the intermediate logic \( C(\{\alpha_i : i \in I\}) \), \( I \subseteq \{0, 1, \ldots\} \) by \( \alpha(I) \). Combining Lemma 2 and 3(iv) we get that \( \alpha_j \in \alpha(I) \) iff \( j \in I \) which gives the following:

**Theorem.**

(i) There exist \( 2^{\aleph_0} \) ordinary \( \Psi \)-intermediate logics (for example the set \( \{\alpha(I) \cap FR_\Psi : I \subseteq \{0, 1, \ldots\}\} \) has the cardinality \( 2^{\aleph_0} \))

(ii) There exist ordinary \( \Psi \)-intermediate logics which are non-finitely axiomatizable;

(iii) For every degree of unsolvability there exists an ordinary \( \Psi \)-intermediate logic with the decision problem of higher degree.

**References**


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