A CRITERION OF FUNCTIONAL COMPLETENESS FOR $B^3$

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The study of the class of the functions $B^3$ corresponding to D. A. Bochvar’s three-valued logic [1] is subject of the present paper. In [2] V. I. Shestakov noticed that $B^3$ can be embedded in the class of the functions corresponding to Łukasiewicz logic $L_3$. In [3], [4] the author examined normal forms for the functions belonging to $B^3$ and the axiomatized algebra corresponding to $B^3$. We make use here of some results and symbolism from [3].

Following A. V. Kuznecov we consider the closure operation $[\ ]$ determined on the subsets of the set $B^3$. Let $K \subseteq B^3$. Then we call $[K]$ the closure of the set $K$. $[K]$ comprises all superpositions ([5]) of functions belonging to $K$. The set of functions $K$ is called closed if $[K] = K$; the set of functions $K \subseteq B^3$ is called pre-complete in $B^3$ provided that $[K] \neq B^3$ and for any function $F \in B^3$ such that $F \notin K$, $[K \cup \{F\}] = B^3$; a set $K$ is said to be functionally complete in $B^3$ if $[K] = B^3$.

Let 0, 1, 2 be the logical values of the logic $B_3$ (0 = falsehood). By $\sim x_1$, $x_1 \cap x_2$, $x_1 \cup x_2$ we denote the functions called: internal negation, internal conjunction, and internal disjunction, respectively [1,3]. Their truth-tables are as follows:

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<th>$x_1 \cap x_2$</th>
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<th>2</th>
<th>$\sim$</th>
<th>$x_1 \cup x_2$</th>
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Let us consider now the function \( J_\alpha x = \begin{cases} 0, & \text{when } x \neq \alpha \\ 2, & \text{when } x = \alpha \end{cases} \) where \( \alpha \in \{0, 1, 2\} \). Let us introduce the following functions of \([1]\): 
\(~x =_{df} J_0 x, \downarrow x =_{df} J_1 x, \vdash x =_{df} J_2 x\). The functions \( \sim x, \uparrow x, x_1 \cap x_2 \) constitute a basis for \( B^3 \) [1].

Let \( F \) be any function from \( B^3 \).

Then \( F(x_1, \ldots, x_n) = J_{x_1} \cup \ldots \cup J_{x_k} \uparrow F(x_1, \ldots, x_n) \) where \( 0 \leq i_k \leq n \), and \( J_{x_j} =_{df} x_j \cap \sim x_j \) (see [3]).

Let us notice that \( \vdash F \) may be presented in the following way: \( \vdash F = G_F \cap K_F \) where \( G_F \) is the part of \( J \)-perfect normal form of \( F \) not containing the occurrences of \( \downarrow x_j \), and \( H_F \) is the part of \( J \)-perfect normal form of \( F \) every conjunctive member of which contains one occurrence of \( \downarrow x_j \) at most.

We denote the set of all functions \( F \in B^3 \) such that \( G_F = 0 \) and \( H_F \neq 0 \) by \( B^3_G \).

By \( G_F \), we denote a function homomorphic to the function \( G_F \).

The following 11 sets of functions are pre-complete in \( B^3 \).

1-5. Let \( B^3_x \), where \( x \) is \( T_0, T_2, S, L, \) or \( M \) be the set of functions \( F \in B^3 \) such that \( G_F \) belongs to \( T_0, T_2, S, L \) or \( M \), respectively, (where \( T_0, T_2, S, L, M \) are pre-complete sets of two-valued functions [6] preserving constant 0, constant 2, self-dual, linear, and monotonic).

6. \( B^3_n = \{ F : \text{there exists } x_i \text{ such that } F(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = 1 \} \)

7. Let us introduce the following notation:

\( X_F = \text{the set of all those variables } x_j \text{ of the function } F \in B^3_n \text{ such that } F(x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n) = 1 \).

Let us take into consideration now those functions \( F \in B^3_{in,imp} \) such that \( G_F = V_F \cap K_F \), where \( V_F = 2 \), and \( K_F \) such that \( K^+_F \neq 0 \) and \( K^-_F \neq 2 \) and \( X_F \cap X_K \neq \emptyset \) and \( X_K \subseteq X_F \).
Let us put:
\[ B^3_Q = \{ F^j : F^j \in B^3_{\text{in,imp}} \text{ and } G^*_F = \text{const} \} \]
\[ B^3_H = \{ H : H \in B^3_{\text{ex}} \text{ and } G^*_H = \text{const} \} \]
\[ B^3_b = \{ F : F \in B^3_{\text{in,imp}} \text{ and } G_F = V_F \cap K_F \text{ and } K_F \neq 2 \text{ and } X_F \cap X_{V_F} = \emptyset \text{ and } X_{K_F} \subseteq X_F \}. \]
The set \( B^3_\gamma = [B^3_{\text{in,p}} \cup B^3_\theta \cup B^3_H \cup B^3_b] \) is pre-complete in \( B^3 \).

8. Let \( x_j \) and \( \neg x_j \), \( j = 1, 2, \ldots \) be, respectively, the variable and its negation in the sense of two-valued logic. Then \( B^3_{p_k} = [B^3_{\text{ex}} \cup \{ F : F \in B^3_{\text{in}} \text{ and } G^*_F = \text{const} \} \cup \{ F^j : \text{there exists } x_j \text{ such that } F^j = J_{x_j} \cup F^j \text{ and } G^*_F \in \{ x_j, \neg x_j \} \} \) is a pre-complete set in \( B^3 \).

9. \( B^3_{p_0} = [B^3_{\text{ex}} \cup (x_1 \cap x_2) \cup \{ F : F \in B^3_{\text{in}} \text{ and } G^*_F = 0 \} \) is pre-complete in \( B^3 \).

10. \( B^3_G = [B^3_{\text{ex}} \cup (x_1 \cup x_2) \cup \{ F : F \in B^3_{\text{in}} \text{ and } G^*_F = 2 \} \) is pre-complete in \( B^3 \).

11. \( B^3_{\text{in,imp,1}} \) is the set of all those functions \( F \) which are non-properly internal and which are of the form \( F = J_{x_j} \cup F \). The set \( B^3_C = [B^3_{\text{ex}} \cup B^3_{\text{in,imp,1}}] \) is pre-complete in \( B^3 \).

**Theorem.** A set \( K \subseteq B^3 \) is functionally complete in \( B^3 \) iff \( K \) is not contained in any of the sets \( B^3_{\text{in},T_0}, B^3_{\text{in},T_2}, B^3_{\text{in},L}, B^3_{\text{in},M}, B^3_{\text{in},T_0}, B^3_{\text{in},S}, B^3_{\text{in},L}, B^3_{\text{in},M}, B^3_{\text{in},\neg M}, B^3_{\text{ex},T_0}, B^3_{\text{ex},T_2}, B^3_{\text{ex},L}, B^3_{\text{ex},M}, B^3_{\text{ex},\neg M}, B^3_{\text{ex},T_0}, B^3_{\text{ex},T_2}, B^3_{\text{ex},L}, B^3_{\text{ex},M}, B^3_{\text{ex},\neg M} \), where \( B^3_{\text{ex},T_0} = [\{ \neg x_1 \cap x_2 \}] \), and \( B^3_{\text{ex},T_2} = [\{ x_1 \cap x_2 \}] \).

**Remark 1.** In [6] V.I Shestakov examined various normal forms of the functions belonging to \( B^3_{\text{ex}} \). It can be proved that: the set of the functions \( N \subseteq B^3 \), is functionally complete in \( B^3 \) iff \( N \) is not contained in the following seven pre-complete sets: \( B^3_{\text{ex},T_0}, B^3_{\text{ex},T_2}, B^3_{\text{ex},L}, B^3_{\text{ex},M}, B^3_{\text{ex},\neg M}, B^3_{\text{ex},T_0}, B^3_{\text{ex},T_2}, B^3_{\text{ex},L}, B^3_{\text{ex},M}, B^3_{\text{ex},\neg M} \), where \( B^3_{\text{ex},T_0} = [\{ \neg x_1 \cap x_2 \}] \), and \( B^3_{\text{ex},T_2} = [\{ x_1 \cap x_2 \}] \).

**Remark 2.** In [7] S. Hallén considered the three-valued logic \( C \) aiming at a systematic study of “nonsense”. Defined connectives of the logic \( C \) are the functions: \( \neg \downarrow x_1, \neg x_1, x_1 \cap x_2 \). It is easy to see that \( B^3_C \subseteq B^3_{\text{ex}} \), where \( B^3_C = [\{ \neg \downarrow x_1, \neg x_1, x_1 \cap x_2 \}] \).
References


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